Linear Algebra and its Applications 437 (2012) 781–782

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

A simple proof of the generalized Craig–Sakamoto theorem

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ARTICLE INFO

Article history:
Received 20 September 2011
Accepted 26 February 2012
Available online 16 April 2012

Submitted by H. Schneider

AMS classification:
15A15
15A18
15A24
15A48

Keywords:
Eigenvalues
Independence
Quadratic forms
Symmetric matrices

ABSTRACT

The Craig–Sakamoto theorem establishes the independence of two quadratic forms in normal variates. In this article, we provide a simple proof of a generalized Craig–Sakamoto theorem.

1. Introduction

The Craig–Sakamoto theorem establishes the independence of two quadratic forms in normal random variables. For its history, refer to [3,7,8]. After some development, the Craig–Sakamoto theorem can be stated as the following mathematical form.

Theorem 1 (Craig–Sakamoto)**.** *Let A and B be $n \times n$ real symmetric matrices. Then $|I_n - xA - yB| = |I_n - xA||I_n - yB|$ for any $x,\ y \in \mathbf{R}$ if and only if $AB = 0$.*

Many proofs can be found in [1,2,4–6,9]. In the next section, we provide a simple proof of a generalized Craig–Sakamoto theorem below.

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0024-3795/$ - see front matter
http://dx.doi.org/10.1016/j.laa.2012.02.028

Theorem 2. *Let A and B be $n \times n$ real symmetric matrices with $a_1, \ldots, a_s$ and $b_1, \ldots, b_t$ as their nonzero eigenvalues. Then the following conditions are equivalent.*

(1) $AB = 0$.
(2) $|I_n - xA - yB| = |I_n - xA||I_n - yB|$ for any $x,\ y \in \mathbf{R}$.
(3) $|I_n - x(A+B)| = |I_n - xA||I_n - xB|$ for any $x \in \mathbf{R}$.
(4) The nonzero eigenvalues of $A + B$ are $\{a_1, \ldots, a_s, b_1, \ldots, b_t\}$.

2. The proof

The proofs for (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are straightforward. To prove (4) $\Rightarrow$ (1), note that the rank $r(A+B) = s+t$ and $|U+V| \geqslant |U| + |V|$ if U and V are positive semi-definite.

Case (a) $s+t = n$: Let $A_1 = \mathrm{diag}(a_1, \ldots, a_s)$ and $B_1 = \mathrm{diag}(b_1, \ldots, b_t)$. Without loss of generality, assume that $A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = T\begin{pmatrix} 0 & 0 \\ 0 & B_1 \end{pmatrix}T'$ and $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ is orthogonal. Then $A + B = \begin{pmatrix} I & T_2 \\ 0 & T_4 \end{pmatrix}\begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}\begin{pmatrix} I & 0 \\ T_2' & T_4' \end{pmatrix}$, $|A+B| = |A_1||B_1||T_4'T_4|$ and $|T_4'T_4| = 1 = |I_t| = |T_2'T_2 + T_4'T_4| \geqslant |T_2'T_2| + |T_4'T_4|$. Thus $T_2 = 0$ and $AB = 0$.

Case (b) $s+t < n$: $r\begin{pmatrix} A \\ B \end{pmatrix} = s+t$ since $r(A+B) \leqslant r\begin{pmatrix} A \\ B \end{pmatrix} \leqslant r(A) + r(B)$.

Let $P = (x_1, x_2, \ldots, x_n)$ be an orthogonal matrix, where $\begin{pmatrix} A \\ B \end{pmatrix} x_i = 0,\ i = 1, 2, \ldots, n-s-t$. Then, $P'AP = \begin{pmatrix} 0 & 0 \\ 0 & A^* \end{pmatrix}$, $P'BP = \begin{pmatrix} 0 & 0 \\ 0 & B^* \end{pmatrix}$ and $P'(A+B)P = \begin{pmatrix} 0 & 0 \\ 0 & A^* + B^* \end{pmatrix}$, where A^* and B^* satisfy (4) and Case (a). Thus $A^*B^* = 0$ and $AB = 0$.

Acknowledgements

This research is supported by the Natural Science Foundation of China (NSFA, Grant No. 1161054). The author thank a referee for helpful comments and suggestions.

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